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Fekete Szego Inequality Alongwith Their Extremal Functions Making Results Sharp For Certain Subclasses Of Analytic Functions

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ABSTRACT: We introduce some classes of analytic functions, its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

KEYWORDS: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

1. Introduction : Let P denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the unit disc ➪ $= \{z: |z| < 1|\}$. Let S be the class of functions of the form (1.1), which are analytic univalent in ➪.

In 1916, Bieber Bach ([1], [2]) proved that $|a_2| \leq 2$ for the functions $f(z) \in S$. In 1923, Löwner [10] proved that $|a_3| \leq 3$ for the functions $f(z) \in S$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and a_2^2 for the class S, Fekete and Szegö [4] used Löwner's method to prove the following well known result for the class S.

Let $f(z) \in S$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, \text{if } \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), \text{if } 0 \leq \mu \leq 1; \\ 4\mu - 3, \text{if} \mu \geq 1. \end{cases}$$

(1.2)

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes S ([3], [9]).

Let us define some subclasses of S.

We denote by S*, the class of univalent starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in P$ and satisfying the condition

$$\operatorname{Re}\left(\frac{zg(z)}{g(z)}\right) > 0, z \in ⇨. \quad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in P$$

and satisfying the condition

$$\operatorname{Re}\frac{((zh^{F}(z))}{h^{F}(z)} > 0, z \in ⇨. \quad (1.4)$$

A function $f(z) \in P$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in ⇨. \quad (1.5)$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \{f(z) \in P; \frac{zf^{F}(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in ⇨\} \quad (1.6)$$

$$\mathcal{K}(A,B) = \{f(z) \in P; \frac{(zf^{F}(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in ⇨\} \quad (1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}(A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as $\{f(z) \in P; (1-\alpha)\left(\frac{zf^{F}(z)}{f(z)}\right)^{\beta} + \alpha\left(\frac{(zf^{F}(z))^{F}}{f'(z)}\right)^{1-\beta} \prec \frac{1+z}{1-z}; z \in ⇨\}$

and we will denote this class as $S^*(f, f', \alpha, \beta)$.

We will deal with two subclasses of $S^*(f, f', \alpha, \beta)$ defined as follows in our next paper:

$$S^*(f, f', \alpha, \beta, A, B) = \{f(z) \in P; (1-\alpha)\left(\frac{zf^{F}(z)}{f(z)}\right)^{\beta} + \alpha\left(\frac{(zf^{F}(z))^{F}}{f'(z)}\right)^{1-\beta} \prec \frac{1+Az}{1+Bz}; z \in ⇨\} \quad (1.8)$$

$$S^*(f, f', \alpha, \beta, \delta) = \{f(z) \in P; (1-\alpha)\left(\frac{zf^{F}(z)}{f(z)}\right)^{\beta} + \alpha\left(\frac{(zf^{F}(z))^{F}}{f'(z)}\right)^{1-\beta} \prec \left(\frac{1+z}{1-z}\right)^{\delta}; z \in ⇨\} \quad (1.9)$$

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in ⇨. Then $f(z)$ is called subordinate to $F(z)$ in ⇨ if there exists a function $w(z)$ analytic in ⇨ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in$ ⇨ and we write $f(z) \prec F(z)$.

By t, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1$. (1.10)

It is known that $|d_1| \le 1, |d_2| \le 1 - |d_1|^2$. (1.11)

2. PRELIMINARY LEMMAS: For $0 < c < 1$, we write $w(z) = (\frac{c+z}{1+cz})$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \cdots. \tag{2.1}$$

3. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in S^*(f, f', \alpha, \beta)$., then

$$|a_3 - \mu a_2^2| \leq$$

$$\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} - 4\mu\right], \text{if} \mu \leq \frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}; \tag{3.1}$$

$$\frac{1}{3\alpha+\beta-4\alpha\beta} \text{if} \frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} \leq \mu \leq \frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}; \tag{3.2}$$

$$\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[4\mu - \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}\right], \text{if } \mu \geq \frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} \tag{3.3}$$

The results are sharp.

Proof: By definition of $S^*(f, f', \alpha, \beta)$, we have

$$(1-\alpha)\left(\frac{zf^F(z)}{f(z)}\right)^{\beta} + \alpha\left(\frac{(zf^F(z))^F}{f'(z)}\right)^{1-\beta} = \frac{1+w(z)}{1-w(z)}; w(z) \in t. \tag{3.4}$$

Expanding the series (3.4), we get

$$(1-\alpha)\{1 + \beta a_2 z + (2\beta a_3 + \frac{\beta(\beta-3)}{2} a_2^2)z^2 + - - -\} + \alpha\{1 + 2(1-\beta)a_2 z + 2(1-\beta)(3a_3-(\beta+2)a_2^2)z^2 + - - -\} = (1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + - - -).$$
(3.5)

Identifying terms in (3.5), we get

$$a_2 = \frac{2}{(1-\alpha)\beta+2\alpha(1-\beta)} c_1 \tag{3.6}$$

$$a_3 = \frac{1}{3\alpha+\beta-4\alpha\beta} c_2 + \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} c_1^2. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{1}{3\alpha+\beta-4\alpha\beta} c_2 + \left[\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2} - \frac{4}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\mu\right] c_1^2. \tag{3.8}$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \le \frac{1}{3\alpha+\beta-4\alpha\beta}|c_2| + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left|\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} - 4\mu\right| |c_1^2|. \qquad (3.9)$$

Using (1.9) in (3.9), we get

$$|a_3 - \mu a_2^2| \le \frac{1}{3\alpha+\beta-4\alpha\beta}(1-|c_1|^2) + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left|\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} - 4\mu\right| |c_1^2|$$

$$= \frac{1}{3\alpha+\beta-4\alpha\beta} + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\left|\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} - 4\mu\right| - \frac{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{3\alpha+\beta-4\alpha\beta}\right]|c_1|^2. \qquad (3.10)$$

Case I: $\mu \le \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$. (3.10) can be rewritten as

$$|a_3 - \mu a_2^2 \le \frac{1}{3\alpha+\beta-4\alpha\beta} + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} - 4\mu\right]|c_1|^2. \qquad (3.11)$$

Subcase I (a): $\mu \le \frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$. Using (1.9), (3.11) becomes

$$|a_3 - \mu a_2^2 \le \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)} - 4\mu\right]. \qquad (3.12)$$

Subcase I (b): $\mu \ge \frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$. We obtain from (3.11)

$$|a_3 - \mu a_2^2 \le \frac{1}{3\alpha+\beta-4\alpha\beta}. \qquad (3.13)$$

Case II: $\mu \ge \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2 \le \frac{1}{3\alpha+\beta-4\alpha\beta} + \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[4\mu - \frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}\right]|c_1|^2. \qquad (3.14)$$

Subcase II (a): $\mu \le \frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$

(3.14) takes the form $|a_3 - \mu a_2^2 \le \frac{1}{3\alpha+\beta-4\alpha\beta}$ (3.15)

Combining subcase I (b) and subcase II (a), we obtain

$$|a_3 - \mu a_2^2| \le \frac{1}{3\alpha+\beta-4\alpha\beta} \; if \; \frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} \le \mu \le \frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} \qquad (3.16)$$

Subcase II (b): $\mu \geq \frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$

Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[4\mu - \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}\right]. \quad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$f_1(z) = (1 + az)^b$

Where $a = \frac{\{(2\alpha+\beta-3\alpha\beta)^2-(1-\alpha)\beta(\beta-3)+4\alpha(1-\beta)(\beta+2)\}a_2^2-4(3\alpha+\beta-4\alpha\beta)a_3}{(2\alpha+\beta-3\alpha\beta)a_2}$

And $b = \frac{(2\alpha+\beta-3\alpha\beta)^2 a_2^2}{\{(2\alpha+\beta-3\alpha\beta)^2-(1-\alpha)\beta(\beta-3)+4\alpha(1-\beta)(\beta+2)\}a_2^2-4(3\alpha+\beta-4\alpha\beta)a_3}$

Extremal function for (3.2) is defined by $f_2(z) = z(1 + Bz^2)^{\frac{A-B}{2B}}$.

Corollary 3.2: Putting $\alpha = 1, \beta = 0$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1-\mu, \text{ if } \mu \leq 1; \\ \frac{1}{3} \text{ if } 1 \leq \mu \leq \frac{4}{3}; \\ \mu - 1, \text{ if } \mu \geq \frac{4}{3} \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

Corollary 3.3: Putting $\alpha = 0, \beta = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3-4\mu, \text{ if } \mu \leq \frac{1}{2}; \\ 1 \text{ if } \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, \text{ if } \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

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